AMYGDALA $z \mapsto z^2+c$

A Newsletter of fractals & M -- the Mandelbrot Set
AMYGDALA, Box 219, San Cristobal, NM 87564

$15.00 for ten issues ($25 overseas)
$15.00 for slide supplement ($25 overseas)
$30.00 for ten issues plus slides ($40 overseas)

Issue #7
November 1, 1987

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MCI MAIL

For those of you that have an account on MCI mail, I would be glad to correspond, take submissions to Amygdala, or answer questions via that medium. My MCI ID is 266-4211, but you can send a letter simply TO: Rollo.

SLIDES (AT LAST!)

Many of you that get the slide supplement have been wondering when you'd start getting it. Well, the first thirteen slides[1] accompany this issue, Amy #7. As a teaser for those of you that don't get the supplement, I'm going to describe the slides here, rather than on a separate sheet.

Andrew LaMance produced a series of forty slides on the Leading Edge model-M with the Sperry HiRes graphics system and 8087 NDP. All images are 320x200 pixels, 256 colors possible per pixel; the dwell limit is 256. The slides form a zoom sequence of the 22-legged Ant, all centered on -0.72398340 + 0.28671980i, with magnifications ranging from 0.714 to 120,000 in a geometric progression. Each one took between 2 minutes and 1 hour to calculate.

Two slides from the ant series are included:
#133: The 22-legged ant just begins to appear.
#137: Here is the ant, resplendent.

LaMance sent along another five slides as well; #141 and #142 are included here:
#141: "Nova": -1.7813235+0.00396121i x 2.5×10^6.
#142: "Love Canal": -0.235125+0.82722i x 24,801.

Slides #163-203 were created by *John Dewey Jones:*
#163: The Julia set of a point in Seahorse Valley: (c = -0.7927 + 0.16089i); z-window is: -3-2i...3+2i.
#164: The basins of attraction for the three roots of $z^3+1 = 0$ as approached by Newton's method.
#175: The Julia set generated by c = -0.7927+0.16089i, dwell limit 1024, collapsed into 256 equivalence classes, range -2.0-1.5i...2.0+1.5i, rendered in garish mauve tones according to preference.
#177: Salmon contrails on purple. "Red Dawn over Key West". False-color satellite photo shows waves of communist influence spreading across the Caribbean to subvert Florida. M-view: -0.1013816-0.9567729i...-0.1012797-0.95639i.
#191: Shows the convergence, and in the case of the yellow and mauve regions, the non-convergence, of the secant method: an iterative solution process similar to the Newton method, defined by:

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)}) \cdot [x^{(k)} - x^{(k-1)}]}{f(x^{(k)}) - f(x^{(k-1)})}$$

This method requires two initial guesses. In the slides, I've applied the method to finding the real solutions of $x^3-1 = 0$, and let the horizontal and vertical axes represent the first and second guess. I've used a dwell limit of 64 and a range of -2-1.5i...2+1.5i. I'm not aware of anyone having reported similar results anywhere. (Though no doubt there are many things I'm not aware of.)
#203: Julia set.

#343-375 were created by *Ken Philip,* (Oct 87).
#343: Handsome red,green,blue five-star from ten-star.
#347: Iron Cross.
#375: Purple & green Indian swastika Julia set.

[1] All slides related to Amygdala business will be numbered for reference purposes: AMY#1 to AMY#∞.

COSMOLOGY AND THE MANDELBROT SET

—*Jonathan Dickau*

My first exposure to the Mandelbrot set was A.K. Dewdney's column in Scientific American. I was deeply impressed by some of the magnified areas detailed so beautifully in the pictures by Peitgen and Richter, but my impression of $\mathcal{M}$ as a whole was probably more significant; I said "Oh, it looks like the Big Bang".

My efforts to recreate some of the wondrous images (on my dad's PC), and later to find shortcuts[1] in the process gave me a tool which illustrates and verifies my observation in a marvelous way. I altered my program for $\mathcal{M}$ to exit the calculation loop when the size of the complex variable is steadily decreasing, (size $[z_k]$ < size$[z_{k-1}]$ < size$[z_{k-2}]$), and color it in by the iteration on which it does so. The picture that is thus formed superimposes a butterfly and a series of circular figures on the image of $\mathcal{M}$ (figure 1). The evolution of these figures around the edge from the cusp appears to mimic the appearance of form that is described by modern cosmological theory (either the 'Big Bang'[2] or perhaps the 'Inflationary Universe' scenario[3]). What I am referring to is the concept of how the elementary particles (making up atoms), nuclei, and atoms precipitated or congealed from the sea of energy which cooled as it expanded to become the universe.

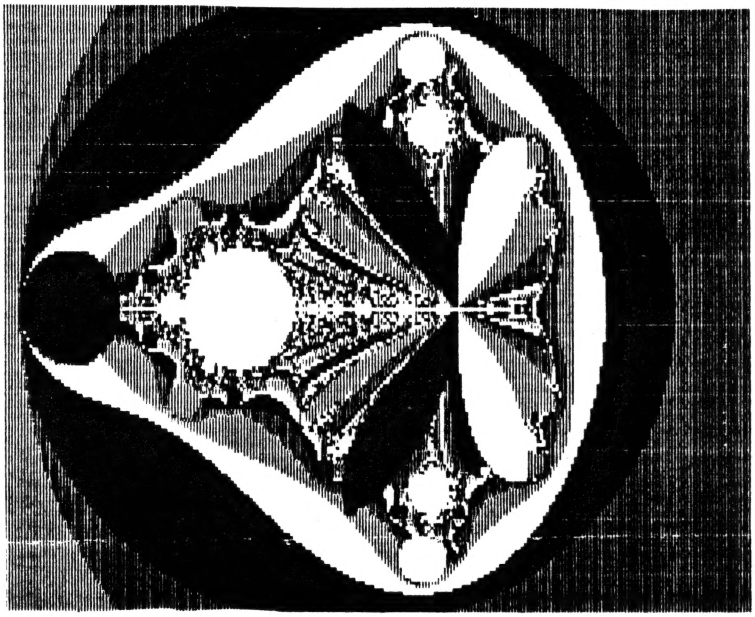

Figure 1 — The Big Bang

If the overall form of the set is seen as similar to the unfoldment of the universe then particular areas should represent a certain type of interaction (i.e. those involving the strong nuclear force, or electromagnetism). A closer look shows precisely this (figure 2); each circular figure being neatly split in two, a graphic representation of the Weak Nuclear force which causes the breakdown of atomic nuclei. Note also that the figures are more nearly circular on the left, and more distorted on the right.

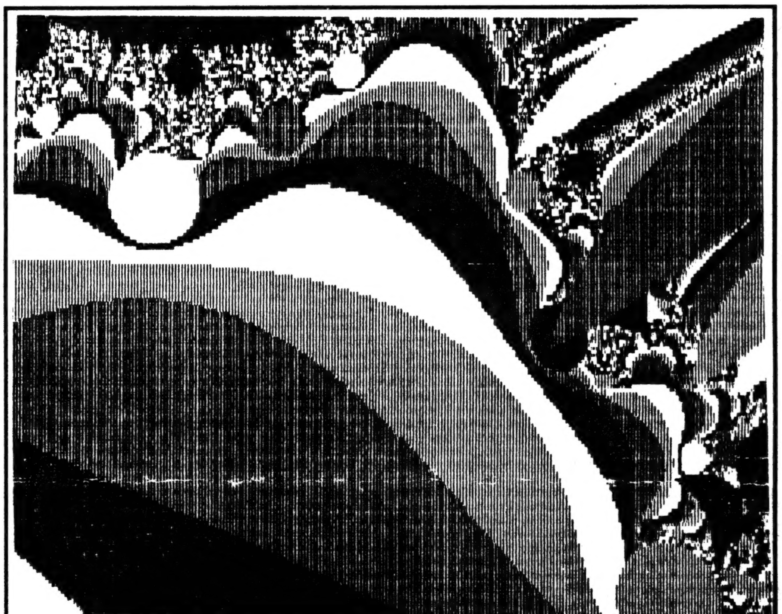

Figure 2 — Interactive Forces

I would love to communicate with any readers who either find these ideas compelling, or somehow insubstantial. Any who have pondered the creation of, or the structure of the universe are welcome to write me. In fact, I hope that those who have not are also interested, as I believe that the resemblance of the Mandelbrot set to the Caduceus of Mercury is not a mere coincidence.

The algorithm used to create these images is outlined here.

```
For each point (A,Bi)
  Set Count = 0  psiz = 0
  Az = A         Bz = B
  Calculate Size
  While size < 2 and
        count < dwell
     Set lsiz = size
     Calc new Az and Bz
     Increment count
     Calculate new size
     If size < lsiz < psiz Then Set size = 3
     Set psiz = lsiz
  Write count to buffer
```

1 With much help from (and thanks to) Mark Little, Glenn Knickerbocker, and Dave Harnett.

2 Alan H. Guth & Paul J. Steinhardt; Scientific American May '84.

3 Joseph Silk; W.H. Freeman & Co. 1980. ISBN 0-7167-1085-4.

Assign color (as desired)
Plot pixel (color)

I can also offer complete software (for IBM PC) on disk for those who don't want to do it themselves.

Jonathan Dickau

MANDELBROT PROGRAMS FOR AMIGA

—*Frank Chambers, Republic of Ireland*

The accompanying table lists the Amiga Mandelbrot generators that I am aware of as of 3 October 87. Those marked with an asterisk are the ones which I have had an opportunity to test. In my opinion #9 is by far the best. However there is bad news. This program, MANDFXP[1], is a demo, available from Tom Granvold, our Amiga Contact, on his disk FSD2. Many of the pull-down menu items are not yet implemented. If you send to the writers for the complete shareware program described in the demo, your letter will be returned marked "moved, left no address". Oh, CygnusSoft, once of 407-1280 Haro Street; Vancouver, BC, CANADA V6E 1EB; where are you? Won't you please contact your fans? The demo is exceedingly fast, but the writeup promises another 25% increase in speed in the enhanced program. If all the menu items were implemented, which they are not in the demo, this would be a truly outstanding program, well worth the requested $15 if you don't send a disk, $10 if you send a disk and mailer.

In view of the current unavailability of ENHANCED MANDFXP, which is second best? My money is on Tom Wilcox's MSE, the Mandelbrot Set Explorer. It has an extended Help file, makes use of pull-down menus, and is reasonably speedy if not fully user-friendly. The French/Mical Version 3.00 would get my next vote, but it is relatively clumsy to use and appears to be in transition between a command-driven and a menu-driven form. The branched HELP file is most irritating, as there is no provision for going back down a branch. You have to fall off the end, as it were, and climb the tree again.

For those who are not familiar with the sources listed in the box: "AMICUS" is a public domain software list which recently contained 20 disks, available from the publishers of the Amiga-only magazine "Amazing Computing"; PiM Publications Inc.; PO Box 869; Fall River, MA 02722, USA. "Fish" is another public domain software list which included 102 disks at last count, also available from PiM or from Fred Fish; 1346 W. 10th Place; Tempe, AZ 85281, USA. The FSD series was described on page 5 of Amygdala #5. When last heard from, Tom Granvold, 1087 C Reed Ave, Sunnyvale, CA 94086, USA, had two disks available. Item 9 is included on disk FSD 2 as well as a good selection of images and a slide show program.

To come: a comparison of ENHANCED MANDXFP and MSE, with a check list suitable for evaluating all Mandelbrot programs, whatever the computer.

[1]Tom Granvold says: On Fish #95 there is a new version of this demo along with a demo of their text editor. The address given in this demo is 1215 Davie St; PO Box 363; Vancouver, BC; CANADA V6E 1N4. I sent a letter to them a couple of weeks ago, but have not yet received a reply. I am putting a copy of this version of the demo on FSD2.

A MONOCHROMATIC THEOREM

In Amy #4 (page 4) I mentioned a "monochromatic theorem" which justifies supposing that if all the points on the perimeter of a rectangle have the same dwell D then all points in its interior have that same dwell.

Since then I've generalized the theorem: it now justifies supposing that if all the points on the boundary of a region — rectangular or otherwise — have dwell in a *range* d...D then all points in its interior have dwell in that same range, where $0 \le d \le D \le \infty$.

I've written up this theorem in a rather formal matter, hoping to publish it in some computer journal. I'm printing it here for those of you who are interested in such matters.

NO	NAME	SOURCE	NOTES
1	mandel	Fish 4	French/Mical, superseded by #2.
2	mandelbrot	Fish 5	French/Mical, superseded by #3.
3	Mandelbrot	Fish 31	*French/Mical, version 3.00.
4	Mandelbrots	Fish 20	Images from three contestants.
5	MSE	Fish 21	*Wilcox's Mandelbrot Set Explorer.
6	Mandelbrot Set	BYTE 12/86	*Schroeder.
7	Mount Mandelbrot	AMICUS 12	3-D view of set.
8	mandelbrot	AMICUS 20	?
9	MANDLE or MANDFXP	CygnusSoft	*Larocque/Dawson, demo (shareware)
10	Mandelbrot	FSD 1	Landrum
11	IMANDELVROOM	FSD 1	Author not known; also on Fish 90
12	Mand	FSD 1	French/Mical (same as #3?
13	(name not known)	?	Mica (mentioned in Amygdala #0a, page 2)

A "MONOCHROMATIC" THEOREM FOR ITERATION OF A QUADRATIC MAP IN A REGION OF THE COMPLEX PLANE

Rollo Silver

INTRODUCTION

There is a great deal of interest in computergraphical representation of the structure of iterations $z \mapsto f(z)$ in the complex plane $\mathbb{C}$ for various functions f. Such iterations are the basis for a variety of colored images of striking beauty and complexity [1], [2].

Given a function f mapping $\mathbb{C}$ into $\mathbb{C}$, the general idea is to consider a finite regular array A of points in a region R of $\mathbb{C}$, and, for each point z in A, to compute the iterates $z \mapsto f(z)$ while a criterion condition P(z) is satisfied. The number of times the iteration is performed on a point z before the criterion becomes unsatisfied is the *dwell* of z. If the criterion is always satisfied, dwell(z) $= \infty$.

The resulting array of dwells is used to create a colored rendition of f over R by assigning a color to each dwell value, and then coloring each point z in A accordingly. Many of these striking color images can be found in [3].

Arthur Cayley [4] was the first to investigate the properties of such iterations by studying the properties of Newton's method in the large for $f(z) = z^2-1$. Gaston Julia [5] and Pierre Fatou [6] investigated the more general case of iteration over rational functions. Interest in the subject has been rekindled in recent times by the work of Benoit Mandelbrot [7].

THE QUADRATIC MAP $z \mapsto z^2+c$

In this paper we concentrate our attention on the quadratic map $f_c(z) = z^2+c$ for z and c in $\mathbb{C}$, considering the iterates $\{z_i\}$ of 0 under f_c:

$$z_0 = 0;$$

$$z_i+1 = f_c(z_i) = z_i^2+c;$$

with the criterion

$$P(z_i) \equiv |z_i| \leq 2.$$

The sequence $\{z_i\}$ is either bounded or not. The set of all c in $\mathbb{C}$ for which the sequence of iterates is bounded is called the *Mandelbrot set,* which we designate here as $\mathcal{M}$.

For computational purposes, the following theorem is basic:

Theorem 1: $\forall z \in \mathbb{C}^{(1)}$, $z \in \mathcal{M}$ if and only if all $|z_i| \leq 2$.

The significance of the criterion $P(z_i) \equiv |z_i| \leq 2$ is a result of this theorem.

The calculation of dwells is time-consuming, hence any relief in the form of significant elimination of those calculations is welcome. Rico Mariani [8] has devised an algorithm for computing views of $\mathcal{M}$ which avoids the time-consuming computation of dwell in the interior of a rectangle R, provided that the dwell is found to be constant for points on the boundary of R.

Mariani's algorithm can be generalized: if the dwell of all points w on the boundary of a rectangle R is confined to an *interval* of values, $d \leq$ dwell(w) $\leq D$, say, then $d \leq$ dwell(z) $\leq D \leq \infty$ for all points z in R; the generalization being based on the following "Monochromatic" theorem:

Theorem 2 (Monochromatic theorem): Let R be a bounded, closed region in the complex plane $\mathbb{C}$ which does not contain all of $\mathcal{M}$. If the dwell of every $z \in \partial R^{(2)}$ satisfies $d \leq$ dwell(z) $\leq D \leq \infty$, then the dwell of every $z \in R$ satisfies it as well.

First, some definitions. A *region* is a nonempty, open, connected set together with some, none, or all of its boundary points. If none of its boundary points are included, the region is an *open* region, or *domain.* If all its boundary points are included, it is a *closed* region. Given a region R, ∂R is the set of all of its boundary points.

The Mandelbrot polynomals, $M_n(z)$, mapping $\mathbb{C}$ into $\mathbb{C}$ are defined, for $n \geq 0$, by:

$$M_0(z) = 0;$$

$$M_{n+1}(z) = M_n(z)^2 + z.$$

Thus:

$$M_1(z) = z,$$

$$M_2(z) = z^2+z,$$

$$M_3(z) = z^4+2z^3+z^2+z,$$

etc. It is obvious that M_n is a polynomial of degree 2^{n-1}.

Definition of dwell: $\forall z \in \mathbb{C}^{(2)}$, *dwell(z)* is the largest n such that $|M_n(z)| \leq 2$. If $|M_n(z)| \leq 2$, $\forall n \geq 0$, then dwell(z) $= \infty$.

To prove Theorem 2 we require four other theorems:

Theorem 3: (Maximum modulus theorem): Let f(z) be a function continuous in the bounded closed region R and analytic in the interior of R. Then there exists a point $w \in \partial R$ such that $\forall z \in R$, $|f(z)| \leq |f(w)|$.

A proof of this theorem will be found in most texts on complex analysis, e.g. [9], page 134.

Theorem 4: (Douady & Hubbard): The Mandelbrot set $\mathcal{M}$ is connected.

This deep theorem is proved in [10].

---

1 "$\forall z \in \mathbb{C}$" means "for all z in $\mathbb{C}$".

2 "$z \in \partial R$" means "z is in (on) the boundary of R".

Theorem 5: If $M_n(z) = 0$ for some $n > 0$, then $z \in \mathcal{M}$.

Proof: Suppose $M_n(z) = 0$, for some $n > 0$. Then

$M_n(z) = M_0(z)$,

$M_{n+1}(z) = M_n(z)^2 + z = z = M_1(z)$,

$M_{n+2}(z) = M_{n+1}(z)^2 + z = M_1(z)^2 + z = M_2(z)$,

and it is easy to show inductively that $M_{n+k}(z) = M_k(z)$, $\forall k \geq 0$. It follows that the $|M_i(z)|$ are bounded (by $s = \max\{|M_k(z)|: 0 \leq k < n\}$), hence $z \in \mathcal{M}$. **Q.E.D.**

Theorem 6: $\forall z \in \mathbf{C}$, $|M_n(z)| \leq 2$ if and only if $n \leq$ dwell(z).

This theorem is proved after Lemma 2, below.

Now to the proof of Theorem 2:

Theorem 2 (Monochromatic theorem): Let R be a bounded, closed region in the complex plane $\mathbf{C}$ which does not contain all of $\mathcal{M}$, and let d and D be integers satisfying $d \leq D \leq \infty$. If the dwell of all $w \in \partial R$ satisfies $d \leq$ dwell(w) $\leq D$, then the dwell of all $z \in R$ satisfies it as well.

Proof: Let R be a bounded, closed region in $\mathbf{C}$ which does not contain all of $\mathcal{M}$, and let $d \leq$ dwell(w) $\leq D \leq \infty$, $\forall w \in \partial R$. We prove that $d \leq$ dwell(z) $\leq D$, $\forall z \in R$.

Case I: $d = \infty$. Then dwell(w) $= \infty$, $\forall w \in \partial R$. We must prove that dwell(z) $= \infty$, $\forall z \in R$. Suppose, contrariwise, that for some $z \in R$, dwell(z) $< \infty$; let $n =$ dwell(z). Then $|M_{n+1}(z)| > 2$ (by Theorem 6), and hence (by Theorem 3) there is some $w \in \partial R$ for which $|M_{n+1}(w)| > 2$, so (again by Theorem 6) $n+1 >$ dwell(w), so dwell(w) $< \infty$, which is a contradiction.

Case II: $d < \infty$. Let I be the interior of R, and K be the complement of R; then both I and K are nonempty open sets: I because R is a region, and K because R is closed and bounded.

By the maximum modulus theorem, there exists a $w \in \partial R$ with $|M_d(z)| \leq |M_d(w)|$, $\forall z \in R$. By hypothesis, $d \leq$ dwell(w), so $|M_d(z)| \leq 2$, by Theorem 6. Therefore $\forall z \in R$, $|M_d(z)| \leq 2$, and so (again by Theorem 6):

$d \leq$ dwell(z), $\forall z \in R$. (1)

If remains to show that dwell(z) $\leq D$, $\forall z \in R$.

If $D = \infty$, it is certainly true that $\forall z \in R$, dwell(z) $\leq D$, and we are done. The case $D < \infty$ remains. In that case (by Theorem 7) no $w \in \partial R$ has $w \in \mathcal{M}$ (since dwell(w) $\leq D < \infty$), therefore $\mathcal{M}$ and ∂R are disjoint. It follows that $\mathcal{M} \subset I \cup K$.

We have by hypothesis that $\mathcal{M}$ is not wholly contained in R, so $\mathcal{M} \cap K \neq 0$. Since I and K are disjoint nonempty open sets, it follows from the connectedness of $\mathcal{M}$ that $\mathcal{M} \subset K$, and $\mathcal{M} \cap I = 0$. Since we just showed that $\mathcal{M} \cap \partial R = 0$, and since $R = I \cup \partial R$, we have $\mathcal{M} \cap R = 0$.

It then follows from Theorem 5 that M_{D+1} has no zeros in R, and hence that $f(z) = 1/M_{D+1}(z)$ exists, is continuous on R, and is analytic in I. Applying the maximum modulus theorem to f, there is a point $W \in \partial R$ such that $\forall z \in R$, $|f(z)| \leq |f(W)|$, i.e. $|M_{D+1}(z)| \geq |M_{D+1}(W)|$, $\forall z \in R$. But dwell(W) $\leq D$, so $||M_{D+1}(W)| > 2$, so $||M_{D+1}(z)| > 2$, and so dwell(z) $\leq D$. Together with (1), this shows that $d \leq$ dwell(z) $\leq D$. **Q.E.D.**

To tidy up, we prove Theorems 6 and 1; but first, two lemmas.

Lemma 1: If $|z| \leq 2$ and $|M_n(z)| = 2+a > 2$, then $|M_{n+k}(z)| \geq 2+2ka$, $\forall k \geq 0$.

Proof: Routine, by induction, as follows:

(1) k=0: $|M_{n+k}(z)| = |M_n(z)| = 2+a = 2+2^k a$.

(2) Suppose $|M_{n+k}(z)| \geq 2+2^k a$ for $k \geq 0$; then $|M_{n+k+1}(z)| = |M_{n+k}(z)^2 + z| \geq ||M_{n+k}(z)^2 - |z|| \geq (2+2^k a)^2 - 2 > 2 + 2^{k+2}a > 2 + 2^{k+1}a$. **Q.E.D.**

Lemma 2: If $|z| = 2+a > 2$, then $|M_k(z)| \geq 2+2^{k-1}a$, ;$k \geq 1$.

Proof: Routine, by induction, as follows:

(1) k=1: $|M_k(z)| = |M_1(z)| = |z| = 2+a = 2+ 2^{k-1}a$.

(2) Suppose $|M_k(z)| \geq 2+2^{k-1}a$ for $k \geq 1$; then $|M_{k+1}(z)| = |M_k(z)^2 + z| \geq ||M_k(z)^2 - |z|| \geq (2+2^{k-1}a)^2 - 2 - a > 2 + (2^{k+1}-1)a = 2 + (2^k+2^k-1)a > 2 + 2^k a$. **Q.E.D.**

Theorem 6: $\forall z \in \mathbf{C}$, $|M_n(z)| \leq 2$ if and only if $n \leq$ dwell(z).

Proof: if: Suppose $n \leq$ dwell(z), but $|M_n(z)| > 2$. Either $|z| \leq 2$ or $|z| > 2$. If $|z| \leq 2$, then by Lemma 1, $|M_{\mathrm{dwell}(z)}(z)| \geq 2+2^{\mathrm{dwell}(z)-n}(|M_n(z)|-2) > 2$, contradicting the definition of dwell. Suppose, on the other hand, that $|z| > 2$. Since $|M_n(z)| > 2$ and $M_0(z) = 0$, $n > 0$, it follows that dwell(z) > 0. Then by Lemma 2, $|M_{\mathrm{dwell}(z)}(z)| \geq 2+2^{\mathrm{dwell}(z)-1}(|z|-2) > 2$, again contradicting the definition of dwell.

only if: Suppose $|M_n(z)| \leq 2$. By definition, dwell(z) is the *largest* k such that $|M_k(z)| \leq 2$, so $n \leq$ dwell(z).

Q.E.D.

Theorem 7: $\forall z \in \mathbf{C}$, $z \in \mathcal{M}$ if and only if dwell(z) = ∞.

Proof: if: Suppose dwell(z) = ∞. Then $\forall n \geq 0$, $|M_n(z)| \leq 2$, by Theorem 6. Since the set $\{|M_n(z)|\}$ is bounded (by 2), $z \in \mathcal{M}$.

only if: Suppose dwell(z) < ∞. Then $|M_{dwell(z)+1}(z)| > 2$. By Lemmas 1 and 2, $\{|M_n(z)|\}$ is unbounded, so $z \notin \mathcal{M}$. **Q.E.D.**

Given the definition of dwell, Theorem 1 is a corollary of Theorem 7:

Theorem 1: $\forall z \in \mathbf{C}$, $z \in \mathcal{M}$ if and only if all $|z_i| \leq 2$.

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TIME AND THE MANDELBROT SET

Ronald A. Lusk

John Dewey Jones has indirectly brought up the question that I have been struggling with since I first became aware of the M Object: Does time exist within the Mandelbrot Set and if so what direction does it flow in?

Mathematically (but I am no mathematician) the answer would be no. The function $f(z) = z^2 + c$ contains no time element — no *t*. Iteration of each point by definition requires time, but that raises the question as to whether the generation of the M Set is the same as exploring the Set as an existent entity. For now, let's say that M exists, temporally independent of the function's calculation.

What makes M so interesting is that we can see it, so maybe we should have a look at what E.N. Namreh calls the Big Picture. A glorious sight in any dimension, and especially intriguing since the boundary of M is supposed to have a dimension somewhere between 1 and 2. A geometrical ghost trapped in the nether world between a line and a surface? Can time exist for a ghost?

I distinctly remember reading that time is a dimension — the fourth dimension in a dynamic three-dimensional system. Einstein assigned time a dimension of one, so if M is to be thought of as a dynamic, temporal entity, then it should have dimension between 2 and 3. Suddenly, the tightly coiled tendrils of the Object are allowed to unspring from their pseudo-planar containment, spiraling outward into the freedom of time. But I am getting ahead of myself.

Dr. Mandelbrot frequently uses a coastline as an example of a fractal. The coast of England is analogous to the boundary of the M Set, each possessing infinite detail of scale. However, I feel there is a more interesting object of comparison: an imaginary map of the Universe.

Imagine the main body of M as our Universe. On M's boundary lie a myriad of self-similar objects, just as in Einstein's expansionary Universe the galaxies lie on the surface of the expanding universal balloon. The analogy is not exact and cannot be since we don't really know how a detached view of the Universe looks; but those larger self-similar objects, such as those at 0°, 180°, and 270°, could be thought of as 'bubbles' holding concentrations of galaxies on their borders. These would be Dr. Mandelbrot's galactic clusters.

This type of imaginary analogous reduction could probably go on for quite a while, limited only by our ability to perceive the extreme scales of our map and the resolving power of our computer. Is this the domain of time? Put another way, does time flow in the cascading movement from scale to scale within M?

Perhaps we should again consult our imaginary map of the Universe. Assume that we have already zoomed in so that we are within our own solar system and looking at Earth. As we gaze we are also travelling within our own envelope of relative time. Language clearly denotes this time travel with verbs: we *gaze,* we *consult,* we *zoom in,* all these actions require our time. So if we zoom in from Earth to, say, England, the transition between scales takes time, relative to our own personal frame of reference. This is the only element of time in this action. We've gone forward in our own reference time, and our map of the Universe *is included* in that reference. The map is a map, a representation of reality; it is not reality itself.

If we were able to actually zoom in on the different scales of

reality, (not a map), then we'd have some truly relativistic troubles as we went. First we would have to decide how we zoom in. From our distant, wide-angle view of the Universe, do we remain stationary while using an infinitely powerful telescope to explore the Universe's details? If so, then we must contend with the observational constraints of the speed of light. Say our viewing point is a billion billion light years away from where the Earth will be. The light we collected to view the Earth would be a billion billion years old. We'd never even find Earth, because the Earth hasn't been formed yet.

We encounter a different problem if we decide to physically move closer to Earth, carrying our infinitely powerful telescope with us. If we move in to a mere billion light years away, we'd see with our telescope the early stages of our planet's formation a billion years ago. Unfortunately, it took us a billion years to get here travelling at .9999 the speed of light.

But here is our temporal element. For every light year we travel towards a specific frame of reference, we perceive that frame of reference as it is one year later (or would it be two years later, since we would be moving opposite to the light?).

After another billion years we would have moved in close enough to see Earth as it is now. We haven't just zoomed in; we've actually moved through space *and* time. It has not only taken time to get to this point; the reference time of Earth has been streaming past us, captured as it is by light.

I think what this means is that if we wish to introduce time into M, we must undrstand that we are not zooming in, but actually moving *through* the Object. To get time out of M, we must pretend to put time in.

It's not difficult to do. Your computer becomes, in effect, not a telescope, but a spaceship flying though the spacetime of the Mandelbrot Set. The speed of your machine would depend on how fast it takes to generate each picture. As Mr. Ulrich Schmidt showed us in Amy #5 by describing his Autobahn-ready Mandelbrot Machine, no one has set a fundamental limit to such computation, yet[1].

The Big Picture does not easily lend itself to imagining the addition of a temporal dimension. It appears quite flat. The task becomes easier when we move close to "shepherd's crooks" or "compound eyes" (Figure 2). Here one can imagine the twists and coils becoming free to spiral through time. I'm afraid I must leave it to other readers to determine if it is possible to really stick time (t) into the formula for the Mandelbrot Set.

A big question that has so far gone unanswered has to do with the nature of the existence of M. Do the details of scale exist before we generate them? I believe the answer depends again on how we think of M.

If we look at the Object and see a flat map with no temporal element, then indeed all we have to do is look closer to perceive the details that such a picture would hold.

If, however, we think of the Object as 'almost' three dimensional with its extra dimension of time as imagined above, then I would have to say that the existence of specific details can only be predicted on the basis of statistical probability. It would be analogous to looking at the real Universe from outside; unless we have absolute knowledge, there's no way we'd be able to say definitely if, where, or when the Earth will exist. However, from that privileged vantage point we would have a lot of information available, perhaps enough information to theorize a formula or two about the nature of the Universe, perhaps even enough to predict the existence of the Earth. But I maintain that we'd never know for sure until we travelled through spacetime and saw for ourselves. So the answer to my question is, it depends on how you conceive of the Mandelbrot Set. Are you sure that the next time you power up your computer, the interesting little detail you looked at yesterday is the same one you're looking at today? So I would say that there is a temporal element to M, but it's optional. Also, it would seem that by contemplating the existence of time in M, an answer has surfaced to John Dewey Jones' timely question, "Does the Mandelbrot Set contain intelligent life?", Yes, Mr. Jones, M contains intelligent life every time we load up a fractal disk and explore. We are the Amygdalartoids.

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[1] C.H. Bennett and R. Landauer, *The Fundamental Physical Limits of Computation*, Scientific American, July 1985.

LETTERS

From Dan Doerer (October 21, 1987):

As I mentioned on the phone, I tried to duplicate the exercise you presented on page 2 of Amygdala #2 (copy enclosed) — mostly to test my understanding of the $z \mapsto z^2+c$ map. My calculations differed from yours even on the first iteration (.7419+.18i vs .7419+.22i). Now I am assured that either there was a slight calculation error in the original or that I am applying the map incorrectly.

My calculations are enclosed. I would appreciate your comments.

RS replies:

You're right about the exercise in Amy #2 — I was wrong.

I just wrote a "C" program to compute the iterates of 0.5+0.09i in double-precision floating point and print them out. Here are the results, transferred directly from the program's output file:

N	ITERATE	MODULUS
0	0.000000+0.000000i	0.00
1	0.500000+0.090000i	0.51
2	0.741900+0.180000i	0.76
3	1.018016+0.357084i	1.08
4	1.408847+0.817034i	1.63
5	1.817304+2.392152i	3.00
6	-1.919795+8.784537i	8.99

Almost the same as your results.

From James E. Loyless (September 24, 1987):
Dear AMY:
This is in response to the "challenge" from Ulrich Schmidt in your July issue. There is a really spectacular picture on the cover of Scientific American's Fall 1987 Book Catalogue which shows a portion of the mu set. Resolution is 4096 x 4096 and iteration limit is 768. It was computed in 250 seconds on the NASA Goddard Lab's MPP (Massively Parallel Processor). By my reckoning, it would take Herr Schmidt over 1500 seconds to compute the nearly 17 million points at that iteration limit, even using single precision. How does one say "touché" in German?
Best regards,
James Elliott Loyless

RS replies:
Very impressive — but the fact that the iteration limit is 768 doesn't justify taking 768 as the average number of iterations per point!

From Max Buscher (March 21, 1987):
I have been "playing with" iterating functions for a couple of years now. Chaos by frequency-doubling, using: f(x) =4 * L * x * (1-x) was my first contact with this area of mathematics. Then I fooled with displaying the orbits of Henon-like functions. And then came the Scientific American article (Aug. '85).

My computer is an APPLE IIc — so plotting the Mandelbrot set or parts of it is always a long story. So my efforts have been involved with speeding things up. I guessed that if an initial seed was a member of the set then the values of x and iy would eventually get "trapped" in a cycle of some length.

At first I wasn't able to exhibit this phenomenon of trapping. I guessed it was related to the number of digits my APPLE was able to process. So I deliberately truncated all numbers to two significant places — like this:
x = (INT(100 * x))/100 for both the real and imaginary parts. And it worked! At least for the entire set, iterated to a maximum depth of 64 iterations. And I looked only for a cycle of four, ie:

    if x(j) = x(j-4) and iy(j) = iy(j-4)
    then exit loop

What's really incredible is that in truncating like this I invoked a catastrophic finite limit on the numbers available — something like 1000 total. This leads me to think that Mandelbrot membership of a seed is determinable by some number-theoretic test.

I tried investigating the relative-primality of x and iy — ie, do x and iy have any common factors — if yes then membership is negative. I can't report any success yet.

I will throw out for your consideration this idea. The algorithm for generating the Mandelbrot set is a partitioning of the reals into two sets — membership and non-membership. These two sets correspond — in a general way — to the rationals and the irrationals. There is a more generalized definition of these two terms (rational and irrational) lurking in here somewhere. The concept of rationality may eventually be broadened to be "context-sensitive", ie, dependent on the domain over which it is defined.

Does anyone there have the time or inclination to comment on any of this? Thanks for listening. No one in my orbit of friends is interested in this stuff and I'm hungry for some communication.
Sincerely,
Max Buscher
129 Magazine Street
Cambridge, MA 02139

RS replies:
Since you read Dewdney's column in the August 1985 *Sci Am,* you probably saw the discussion toward the end of it about the behavior of the *integer* iteration $n \mapsto n^2+a$ (mod 100). This is closely related to your truncation process. To go from this to thinking that "Mandelbrot membership of a seed is determinable by some number-theoretic test", though, seems to be a leap of prodigious audacity. Good luck!

Your conjecture about "trapping" is incorrect. There certainly are points in $\mathcal{M}$ which cycle (e.g. 0, -2), but there are others which do not, e.g. 1/4, whose iterates increase, converging to 1/2 from below: $(1/2-\varepsilon)^2 + 1/4 = 1/2-\varepsilon+\varepsilon^2$, which is between $1/2-\varepsilon$ and $1/2$, for $0 < \varepsilon < 1$.

My immediate response to your suggestion of an analogy $\mathcal{M}$/non-$\mathcal{M}$ :: rationals/irrationals is interest, tempered with skepticism. For one thing, the rationals and irrationals are dense in each other, while $\mathcal{M}$ and non-$\mathcal{M}$ are definitely not.

Well, I guess this letter answers your question "Does anyone there have the time or inclination to comment on any of this?" Please keep on thinking creatively about these matters, and don't let the slight negative slant of my comments discourage you!

CIRCULATION

As of November 1, 1987 Amygdala has 180 paid-up subscribers, of which 53 have the supplemental color slide subscription.